

# On a class of determinants

Milan Janjić \*

## Abstract

A class of determinants is introduced. Different kind of mathematical objects, such as Fibonacci, Lucas, Tchebychev, Hermite, Laguerre, Legendre polynomials, sums and covergents are represented as determinants from this class.

A closed formula in which arbitrary term of a homogenous linear recurrence equation is expressed in terms of the initial conditions and the coefficients is proved.

## 1 Introduction

As a particular case of Theorem 2.1 which is proved in [2] we have the following theorem.

**Theorem 1.1.** *Let  $a_1, a_2, \dots$  be a sequence which terms are from a commutative ring  $R$ . Assume that*

$$a_{k+1} = \sum_{i=1}^k p_{k,i} a_i, \quad (p_{k,i} \in R, i = 1, \dots, k, k = 1, \dots) \quad (1)$$

Then

$$a_{k+1} = a_1 \begin{vmatrix} p_{1,1} & p_{2,1} & p_{3,1} & \cdots & p_{k-1,1} & p_{k,1} \\ -1 & p_{2,2} & p_{3,2} & \cdots & p_{k-1,2} & p_{k,2} \\ 0 & -1 & p_{3,3} & \cdots & p_{k-1,3} & p_{k,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{k-1,k-1} & p_{k-1,k} \\ 0 & 0 & 0 & \cdots & -1 & p_{k,k} \end{vmatrix}.$$

## 2 Some consequences

Several important mathematical objects may be represented in the form of a determinant of the above form.

---

\*Department of Mathematics and Informatics, University of Banja Luka, Republic of Srpska

First of all, it is the case with the sequence of natural numbers, which may be defined by the following recurrence relation:

$$a_1 = 1, a_{1+k} = a_1 + a_k, (k = 1, 2, \dots).$$

We thus obtain the following proposition.

**Proposition 2.1.** *Let  $n$  be arbitrary natural number. Then*

$$n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & \cdots & 1 & 0 \\ 0 & 0 & \vdots & \cdots & -1 & 1 \end{vmatrix},$$

where the size of the determinant is  $n$ .

Polynomials are also formed by the rule (1). Namely, take a sequence  $p_0, p_1, \dots$  of elements of  $R$ . Define coefficients in (1) in the following way.

$$p_{k,1} = p_{k-1}, p_{k,k-1} = x, p_{k,i} = 0. (i \neq 1, k-1).$$

Then the formula (1) becomes

$$1, p_0, a_{1+k} = p_{k-1}a_1 + xa_{k-1}, (k = 2, \dots, n)$$

and obviously  $a_{n+1} = f_n(x) = p_0x^n + p_1x^{n-1} + \dots + p_0$ . We thus obtain the following proposition.

**Proposition 2.2.** *Let  $f_n(x) = p_0x^n + p_1x^{n-1} + \dots + p_0$  be a polynomial from  $R[x]$ . Then*

$$f_n(x) = \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-1} & p_n \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & \cdots & x & 0 \\ 0 & 0 & \vdots & \cdots & -1 & x \end{vmatrix}.$$

Partial sums of a series may also be represented as determinants. Taking, in particular,  $x = 1$  in the preceding equation we obtain  $f_n(1) = \sum_{i=0}^n p_i$ , that is,

$$\sum_{i=0}^n p_i = \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-1} & p_n \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & \cdots & 1 & 0 \\ 0 & 0 & \vdots & \cdots & -1 & 1 \end{vmatrix}.$$

Some important classes of polynomials are given by recurrence relations. Applying Theorem 1.1 we may represent them as tridiagonal determinants. This representation seems to be different from well-known representation of orthonormal polynomials by Jacobi determinants.

Taking

$$a_1 = 1, a_2 = x, a_{k+1} = a_{k-1} + xa_k,$$

Theorem 1.1 yields the following proposition.

**Proposition 2.3.** *If  $F_1(x) = 1$ ,  $F_2(x) = x$ ,  $F_3(x), \dots$  are Fibonacci polynomials then*

$$F_{n+1}(x) = \begin{vmatrix} x & 1 & 0 & \cdots & 0 & 0 \\ -1 & x & 1 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix}. \quad (2)$$

The size of the determinant is  $n$ .

Taking additionally  $x = 1$  we obtain the well-known formula for Fibonacci numbers.

$$F_{n+1} = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix}, \quad (3)$$

with  $F_1 = F_2 = 1$ .

Under the conditions

$$a_1 = 1, a_2 = x, a_3 = 2a_1 + xa_2, a_{k+1} = a_{k-1} + xa_k, (k > 2),$$

Theorem 1.1 produces the following equation for Lucas polynomials  $L_n(x)$ .

$$L_{n+1}(x) = \begin{vmatrix} x & 2 & 0 & \cdots & 0 & 0 \\ -1 & x & 1 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix}.$$

The size of the determinant is  $n$ .

The well-known recurrence relation for Tchebychev polynomials  $T_k(x)$  of the first kind is:

$$T_0(x) = 1, T_1(x) = x, T_k(x) = -T_{k-2}(x) + 2xT_{k-1}(x), (k > 2).$$

Theorem 1.1 implies the following proposition.

**Proposition 2.4.** For Tchebychev polynomials of the first kind  $T_k(x)$  we have

$$T_k(x) = \begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2x & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{vmatrix}.$$

In the same way we obtain the following proposition.

**Proposition 2.5.** For Tchebychev polynomials of the second kind  $U_k(x)$  we have

$$U_k(x) = \begin{vmatrix} 2x & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2x & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{vmatrix}.$$

**Note 2.6.** The preceding classes of polynomials satisfy linear homogenous recurrence equation with constant coefficients.

The next classes also satisfy homogenous linear recurrence equations, but not with constant coefficients.

For Hermite polynomials  $H_n(x)$  we have the following recurrence relation.

$$H_0(x) = 1, H_1(x) = 2x, H_{n+1}(x) = -2nH_{n-1}(x) + 2xH_n(x), (n \geq 2).$$

Applying Theorem 1.1 for  $n = 1, 2, \dots$  we obtain the following proposition.

**Proposition 2.7.** For Hermite polynomials  $H_n(x)$  we have

$$H_n(x) = \begin{vmatrix} 2x & -2 & 0 & \cdots & 0 & 0 \\ -1 & 2x & -4 & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & -2(n-1) \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{vmatrix}. \quad (4)$$

The recurrence relation for Legendre polynomials is:

$$P_0(x) = 1, P_1(x) = x, P_{n+1}(x) = -\frac{n}{n+1}P_{n-1}(x) + \frac{2n+1}{n+1}xP_n(x), (n \geq 2).$$

Hence, the following proposition holds.

**Proposition 2.8.** *For Legendre polynomials  $P_k(x)$  we have*

$$P_n(x) = \begin{vmatrix} x & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -1 & \frac{3}{2}x & -\frac{2}{3} & \cdots & 0 & 0 \\ 0 & -1 & \frac{5}{3}x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{2n-3}{n-1}x & -\frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & -1 & \frac{2n-1}{n}x \end{vmatrix}.$$

In the same way we have the following proposition

**Proposition 2.9.** *For Laguerre polynomials we have*

$$L_n(x) = \begin{vmatrix} 1-x & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -1 & \frac{3-x}{2} & -\frac{2}{3} & \cdots & 0 & 0 \\ 0 & -1 & \frac{5-x}{3}x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{2n-3-x}{n-1} & -\frac{n-1}{n} \\ 0 & 0 & 0 & \cdots & -1 & \frac{2n-1-x}{n} \end{vmatrix}.$$

Taking  $p_{k,k} = p_k$ ,  $p_{k,k-1} = 1$  in (1) we obtain

$$a_1 = 1, p_1, a_{1+k} = a_{k-1} + p_k a_k, (k = 2, \dots).$$

Terms of this sequence are convergents of  $p_1, p_2, \dots, p_n, \dots$  and are denoted by  $(p_1, p_2, \dots, p_n)$ . We thus have the following well-known formula.

$$(p_1, p_2, \dots, p_n) = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & p_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & -1 & p_n \end{vmatrix}. \quad (5)$$

Our next result is a closed formula in which arbitrary term of a homogenous linear recurrence equation is expressed in terms of initial conditions and coefficients of the equation. Such one formula is usually obtained by the calculation of powers of a matrix, as in [1, p. 62].

If  $m$  is a fixed natural number, and  $a(1), a(2), \dots, a(m)$  arbitrary elements of  $R$ . Consider a homogenous linear recurrence equation of the form:

$$a(k) = p_1(k)a(k-m) + p_2(k)a(k-m-1) + \cdots + p_m(k)a(k-1), (k > m). \quad (6)$$

According to Theorem 1.1 we have the following theorem.

**Theorem 2.10.** *Let  $m$  be a fixed natural number,  $a(1), a(2), \dots, a(m)$  arbitrary elements of  $R$ , and let  $a(k)$ , ( $k > m$ ) be as in (6). Then*

$$a(k) = \begin{vmatrix} a(1) & a(2) & \cdots & a(m) & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & p_1(k) & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & p_2(k) & p_1(k) & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & p_m(k) & p_{m-1}(k) & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & p_m(k) & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & p_m(k) & \cdots & p_1(k) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & \cdots & p_2(k) & p_1(k) \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & p_m(k) & p_{m-1}(k) \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & p_m(k) \end{vmatrix}.$$

The size of the determinant is  $k$ .

We shall illustrate this formula by an example arising in differential equations.

Consider the equation

$$(x+1)y'' + y' + xy = 0, \quad (y(0) = 1, y'(0) = 0).$$

Let  $y(x) = \sum_{k=0}^{\infty} u(k)x^k$  be the solution. For coefficients  $u(k)$  we easily obtain the following recurrence relation.

$$u(k+2) = -\frac{k+1}{k+2}u(k+1) - \frac{1}{(k+1)(k+2)}u(k-1), \quad (k \geq 0),$$

with

$$u(0) = 1, \quad u(1) = 0, \quad u(2) = 0.$$

Applying Theorem 2.10 for  $u(k)$ , ( $k > 2$ ) we obtain

$$u(k) = \begin{vmatrix} 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & -\frac{k-1}{k} & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & -\frac{k-1}{k} & \cdots & 0 \\ 0 & 0 & -1 & -\frac{1}{(k-1)k} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & -\frac{1}{(k-1)k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k-1}{k} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{(k-1)k} \end{vmatrix}.$$

## References

- [1] R. P. Agarwal, 'Difference equations and inequalities ', *M. Dekker, 2000*
- [2] M. Janjić , 'Some results on determinants ', *to appear*